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PRONORMAL SUBGROUPS AND HOMOMORPHS IN FINITE GROUPS

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ABSTRACT

A local version of the theory of homomorphs and Schunck classes is given. It is shown that for any finite soluble group the pronormal subgroups are precisely the covering subgroups with respect to "Schunck sets" in this group. As an application simple proofs of some results on pronormal subgroups of finite soluble groups are obtained. Finally a question of Doerk is answered in the negative: any finite soluble group is a subgroup of a minimal non-trivial pronormal subgroup of some finite soluble group.

Introduction

Various attempts have been made to give a so-called "local" version of the theory of Schunck classes and (saturated) formations of finite soluble groups (cf. [13, 10, 9, 15, 12]); that is, to define the notions of homomorph and covering subgroup with respect to a single group (rather than classes of groups). In §1 of the present note we exhibit a criterion for existence of covering subgroups, which is both necessary and sufficient,[††] and a corresponding result on conjugacy of covering subgroups. This will be applied (in §2) to show that a subgroup of a finite soluble group G is pronormal if, and only if, it is a covering subgroup with respect to some G-homomorph. As an application we get simple proofs of some basic results on pronormal subgroups.

In §3 we shall show that every finite soluble group can be embedded in a minimal non-trivial pronormal subgroup of some finite soluble group.

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[††] Despite the claim in [14], p. 184, this aim has not been achieved in [12].

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1. A local theory of homomorphs

Unless stated otherwise, in this section G shall always denote an arbitrary finite group.

1.1. DEFINITIONS. (a) $\mathrm{Sec}(G)=\{X/Y \mid Y \trianglelefteq X \leqq G\}$ is the set of all *sections* of G; $\mathrm{Sec}_1(G)=\{X/X \mid X \leqq G\}$.

(b) Let $\mathscr{X} \subseteq \mathrm{Sec}(G)$, $\mathscr{Y} \subseteq \mathrm{Sec}(G)\setminus \mathrm{Sec}_1(G)$. Then

$$\mathrm{hom}_G(\mathscr{X})=\{X^gN/Y^gN \mid X/Y \in \mathscr{X},\ g\in G,\ N \leqq G,\ X^g \leqq N_G(N)\} \cup \mathrm{Sec}_1(G),$$

$$\mathrm{min}_G(\mathscr{Y})=\{X^g/Y^g \mid X/Y\in \mathscr{Y},\ g\in G,\ \mathrm{hom}_G(\{X/Y\})\cap \mathscr{Y} \subseteq \{X^h/Y^h \mid h\in G\}\}$$

and $\mathscr{X}$, $\mathscr{Y}$ is called a *G-homomorph*, *G-boundary* if, respectively, $\mathscr{X}=\mathrm{hom}_G(\mathscr{X})$, $\mathscr{Y}=\mathrm{min}_G(\mathscr{Y})$.

Further, we put

$$h_G(\mathscr{Y})=\{X/Y\in \mathrm{Sec}(G) \mid \mathrm{hom}_G(\{X/Y\})\cap \mathscr{Y}=\emptyset\},$$

$$b_G(\mathscr{X})=\mathrm{min}_G(\mathrm{Sec}(G)\setminus \mathscr{X}), \quad \text{provided that } \mathrm{Sec}_1(G)\subseteq \mathscr{X};$$

it is easy to see that both $\mathrm{hom}_G(\mathscr{X})$ *and* $h_G(\mathscr{Y})$ *are G-homomorphs,* $\mathrm{min}_G(\mathscr{Y})$ *is a G-boundary, and so is* $b_G(\mathscr{X})$ *for all* $\mathscr{X}$ *containing* $\mathrm{Sec}_1(G)$.

(In what follows we will normally suppress the subscript G in hom_G, min_G, h_G, b_G, unless several groups are under consideration.)

The proof of our first result is similar to the proofs of [1], 1.2, and [4], 2.1.

1.2. PROPOSITION. (a) b_G *and* h_G *induce mutually inverse bijections between the set of all G-homomorphs and the set of all G-boundaries.*

(b) *Let* $X/Y\in \mathrm{Sec}(G)$ *and* $\mathscr{H}=\mathrm{hom}_G(\mathscr{H})$. *Then* $X/Y\notin \mathscr{H}$ *iff* $XN/YN \in b_G(\mathscr{H})$ *for some* $N\leqq G$ *such that* $X\leqq N_G(N)$.

(c) *Let* $\mathscr{H}=\mathrm{hom}_G(\mathscr{H})$, *and suppose that* $\mathscr{R}\subseteq \mathrm{Sec}(G)$ *is such that* $\mathscr{R}^g=\mathscr{R}$ *for every* $g\in G$. *Then* $b_G(\mathscr{H})\subseteq \mathscr{R}$ *iff for each* $X/Y\in \mathrm{Sec}(G)$ *we have* $X/Y\in\mathscr{H}$ *whenever* $\mathrm{hom}_G(X/Y)\cap \mathscr{R}\subseteq \mathscr{H}$.

1.3. DEFINITIONS. (a) Let $\mathscr{H}$ be a G-homomorph. Then $H\leqq G$ is called an *$\mathscr{H}$-covering subgroup of G* ($H\in \mathrm{Cov}_{\mathscr{H}}(G)$) if $H\in\mathscr{H}$ and $X=HY$ whenever $Y\trianglelefteq X\leqq G$ satisfy $H\leqq X$ and $X/Y\in\mathscr{H}$. If for all $N\trianglelefteq G$, HN/N is in $\mathscr{H}$ and is maximal among the subgroups of G/N belonging to $\mathscr{H}$, then H is called an *$\mathscr{H}$-projector of* $G(H\in \mathrm{Proj}_{\mathscr{H}}(G))$. Notice that $\mathrm{Cov}_{\mathscr{H}}(G)=\{H\leqq G \mid H\leqq U\leqq G \Rightarrow H\in \mathrm{Proj}_{\mathscr{H}}(U)\}$.

(b) A G-homomorph $\mathscr{H}$ is called *primitive-closed* if $X/Y\in\mathrm{Sec}(G)$ is in $\mathscr{H}$

whenever all X/Z such that $Y \leqq Z \trianglelefteq X$ which are primitive groups (in the sense of [4], 1.1) are in $\mathscr{H}$.

$\mathscr{H}$ is said to be *induced by subgroups* (of G) if there exists a set $\mathscr{H}_0$ of subgroups of G such that $\mathscr{H} = \mathrm{hom}_G(\mathscr{H}_0)$.

In view of the first main result of [4] (namely, 4.2) one might be tempted to conjecture that the primitive-closed G-homomorphs (which are induced by subgroups) are precisely the G-homomorphs with the property $\mathrm{Proj}_{\mathscr{H}}(U) \neq \emptyset$ for each $U \leqq G$; both these conditions are of course consequences of the latter property. The following examples, however, will show that one needs stronger hypotheses.

1.4. EXAMPLES. (a) Let A and B be isomorphic non-abelian simple groups and put $G = A \times B$, $\mathscr{H} = \mathrm{hom}_G(\mathscr{H}_0)$, where $\mathscr{H}_0 = \{A, B\}$. Then $G \in b_G(\mathscr{H})$, which consists of primitive groups (i.e., $\mathscr{H}$ is a primitive-closed homomorph — see 1.2c), but $\mathrm{Proj}_{\mathscr{H}}(G) = \emptyset$. (Compare with [4], 4.1b + c.)

(b) Let H be a group with an irreducible, faithful module M over some prime field GF(p) such that the semidirect product $G_0 = HM$ has a subgroup K complementing M in G_0 which is not G_0-conjugate to H (i.e., $H^1(H, M) \neq 0$). Further let C be any group of prime order. Consider the direct product $G = G_0 \times C$ and the G-homomorph $\mathscr{H} = \mathrm{hom}_G(\mathscr{H}_0)$, where $\mathscr{H}_0 = \{H, KC\}$. Again $b_G(\mathscr{H})$ consists of primitive groups. It is easy to see that $H \in \mathrm{Proj}_{\mathscr{H}}(HC)$, $HC/C \in \mathrm{Proj}_{\mathscr{H}}(G/C)$, but $H \notin \mathrm{Proj}_{\mathscr{H}}(G)$: in fact, $HM/M < G/M = (KC)M/M \in \mathscr{H}$.

This example shows that the property which has been most crucial in our proof of existence of projectors for Schunck classes of finite groups (see [4], §§3, 4) is no longer available in the present context, even in case when $b_G(\mathscr{H})$ consists of primitive groups. (Note that in this example $KC \in \mathrm{Proj}_{\mathscr{H}}(G) \neq \emptyset$.)

(c) To get an example as in (b), but with $\mathrm{Proj}_{\mathscr{H}}(G) = \emptyset$, we put $G = G_0 \times E$, where G_0 is as in (b) and E is a non-abelian simple group. We may choose E such that, firstly, $E \cong H \cong K$, and secondly, E has a Sylow q-subgroup C of prime order $q \neq p$. Now let $\mathscr{H}$ be the G-homomorph generated by $\{H, KC, E\}$. It is a routine matter to check that $\mathscr{H}$ is primitive-closed, but $\mathrm{Proj}_{\mathscr{H}}(G) = \emptyset$.

(The reader should note that the last example is a combination of the preceding ones: we have not been able to produce an example of a primitive-closed G-homomorph $\mathscr{H}$ induced by subgroups with $\mathrm{Proj}_{\mathscr{H}}(G) = \emptyset$, but $\mathrm{Proj}_{\mathscr{H}}(X/Y) \neq \emptyset$ for every $X/Y \in \bigcup \{b_U(\mathscr{H} \cap \mathrm{Sec}(U)) \mid U \leqq G\}$ — yet one would expect examples of this sort to exist.)

(d) Let $G = X \times Y$ with $X \cong Y$ of order 2, and let Z be the third subgroup of

G of order 2. Then $\mathcal{H} = \hom_G(\{Z\}) = \{Z, G/X, G/Y\} \cup \mathrm{Sec}_1(G)$ has the following properties:

$$\mathrm{Proj}_{\mathcal{H}}(G) = \{Z\} \quad \text{and} \quad \mathcal{H} \not\ni X \cong Z \cong G/Z = XZ/Z \notin \mathcal{H}$$

(in fact, $G/Z \in b_G(\mathcal{H})$);

$$\mathcal{H} \cap \mathrm{Sec}(X) = \{1, X/X\}, \quad \mathrm{Proj}_{\mathcal{H}}(A/B) = \{B/B\} \qquad \text{for every } A/B \in \mathrm{Sec}(X),$$

and

$$b_X(\mathcal{H} \cap \mathrm{Sec}(X)) = \{X\} \neq b_G(\mathcal{H}) \cap \mathrm{Sec}(X) = \{G/Z\} \cap \mathrm{Sec}(X) = \emptyset.$$

The last example, together with 2.1 below, provides justification for not including in our definition of G-homomorph the condition

$$(\#) \qquad U \leqq N_G(N), \quad UN/N \in \mathcal{H} \Rightarrow U/U \cap N \in \mathcal{H}.$$

For soluble groups we have the following local version of the main theorems of Schunck's [11]:

1.5. THEOREM. *Let $\mathcal{H}$ be a G-homomorph. Consider the following two statements*[†]:

(i) $\mathrm{Cov}_{\mathcal{H}}(U) \neq \emptyset$ *for each* $U \leqq G$.

(ii) $\mathcal{H}$ *is primitive-closed and induced by subgroups.*

(a) (i) *implies* (ii).

(b) *Suppose that $\mathcal{H}$ fulfills condition* (ii). *If G is soluble, then*

$$(*) \qquad \begin{array}{l} H \leqq U \leqq G, \quad N \trianglelefteq U, \quad H \in \mathrm{Cov}_{\mathcal{H}}(HN), \\ HN/N \in \mathrm{Cov}_{\mathcal{H}}(U/N) \Rightarrow H \in \mathrm{Cov}_{\mathcal{H}}(U). \end{array}$$

(c) (ii) *and* $(*)$ *imply* (i); *in particular, for soluble G statements* (i) *and* (ii) *are equivalent.*

(d) *If G is soluble, then* $\mathrm{Cov}_{\mathcal{H}}(G)$ *is a set of G-conjugate subgroups of G.*

PROOF. (a) An argument as in [4], 4.1a, utilizing 1.2c, yields primitive closure of $\mathcal{H}$. Further, if $X/Y \in \mathcal{H}$, then $X = HY$ for some $H \in \mathrm{Cov}_{\mathcal{H}}(X)$, and so $\mathcal{H}$ is induced by subgroups.

(b) We shall proceed by induction on $|U|$ and may therefore suppose that N is minimal normal in U. Let $H \leqq X \leqq U$, $Y \trianglelefteq X$, $X/Y \in \mathcal{H}$; we have to show

[†] Here and in what follows we use the following notations: if $U \leqq G$, then $\mathrm{Cov}_{\mathcal{H}}(U)$ means $\mathrm{Cov}_{\mathcal{H} \cap \mathrm{Sec}(U)}(U)$; note that $\mathcal{H} \cap \mathrm{Sec}(U)$ is a U-homomorph for each G-homomorph $\mathcal{H}$. Moreover, the reader will not object to our writing $\mathrm{Cov}_{\mathcal{H}}(G/N)$ ($N \trianglelefteq G$, $\mathcal{H}$ a G-homomorph), which, strictly speaking, is not defined.

that $X = HY$. Since XN inherits from U the hypothesis in (*), we may assume that $XN = U$. Hence either X is a maximal subgroup of U complementing the abelian minimal normal subgroup N, or $X = U$.

First suppose that $X < U$. From $X/Y \in \mathscr{H}$ we get that $U/YN = XN/YN \in \mathscr{H}$. Consequently, $HN/N \in \mathrm{Cov}_{\mathscr{H}}(U/N)$ covers U/YN, i.e., $U = HYN$. Dedekind's identity yields the desired conclusion:

$$X = HYN \cap X = HY(N \cap X) = HY.$$

Now let $X = U$. By the way of contradiction, let $HY < U$. As in the previous paragraph HN/N covers U/YN:

$$U = HYN.$$

Hence HY is a maximal subgroup of U complementing N. W.l.o.g., $Y = \mathrm{Core}_U(HY)$. We cannot possibly have that $HY \in \mathscr{H}$, for then $U/N = (HY)N/N$ should be in $\mathscr{H}$, too, $HN/N \in \mathrm{Cov}_{\mathscr{H}}(U/N)$ should coincide with U/N, whence $H \in \mathrm{Cov}_{\mathscr{H}}(HN)$ should cover U/Y. In particular, $Y \neq 1$. Moreover, application of 1.2b yields a section $HY/Z \in b_{HY}(\mathscr{H})$. Put $T = Z \cap Y \trianglelefteq (HY)N = U$: note that $Y \trianglelefteq U$ centralises N.

Suppose that $T \neq 1$. Clearly, HT/T is a proper subgroup of HY/T, and HTN is a proper subgroup of U. The inductive hypothesis shows that, firstly, $H \in \mathrm{Cov}_{\mathscr{H}}(HTN)$ (and thus $HT/T \in \mathrm{Cov}_{\mathscr{H}}(HNT/T)$), and secondly, $HT/T \in \mathrm{Cov}_{\mathscr{H}}(U/T)$. From this a contradiction emerges: HT/T covers $(U/T)/(Y/T)$.

Hence $T = 1$; more precisely, we have shown that $Z \cap Y = 1$ for each $Z \trianglelefteq HY$ such that $HY/Z \in b_{HY}(\mathscr{H})$. Given one such Z (the existence of which has been shown already), we may take $Z_0/Z \leqq YZ/Z \neq 1$ minimal normal in HY/Z and consider $S = Z_0 \cap Y$, a minimal normal subgroup of HY and U. Then our argument says that HY/S cannot have a quotient in $b_{HY/S}(\mathscr{H}) \subseteq b_{HY}(\mathscr{H})$, and so is in $\mathscr{H}$ (cf. 1.2b). Also, $U/SN = (HY)N/SN \in \mathscr{H}$. Therefore, if $S \leqq R \trianglelefteq U$ and $U/R \in b_U(\mathscr{H})$, then $N \not\leqq R$. Hence NR/R is the unique minimal normal subgroup of the primitive group U/R: note that $\mathscr{H}$ is primitive-closed. Further, $(HY)R/R \in \mathscr{H}$ supplements and thus complements NR/R: indeed, $HY/S = HS/S \in \mathscr{H}$ is a consequence of $U/SN \in \mathscr{H}$, $U = (HY)N$, $(HY) \cap N = 1$, and $HN/N \in \mathrm{Cov}_{\mathscr{H}}(U/N)$. Now both HY and HYR are complements of N in U, and must coincide. That is to say, $R \leqq C_{HY}(N) = \mathrm{Core}_U(HY) = Y$. Both U/Y and U/R are primitive groups with minimal normal subgroup U-isomorphic to N, whence $R \leqq Y$ yields a contradiction as follows: $R = Y$, $\mathscr{H} \ni U/Y = U/R \in b_U(\mathscr{H})$. We have shown:

$U/S \in \mathscr{H}$ for every minimal normal subgroup $S \leqq Y$ of U; and

HN is a maximal subgroup of U complementing S in U.

Choose one such S. Then there exists $K \leqq U$, $K \in \mathscr{H}$, such that $U = KS$, i.e., K is a maximal subgroup complementing S in U. We also have that $N \leqq K$, for otherwise $U/N = KN/N \in \mathscr{H}$ should be covered by $HN/N \in \mathrm{Cov}_{\mathscr{H}}(U/N)$. The same argument shows that $U/N \notin \mathscr{H}$, and with an argument as above, considering $N \leqq V \trianglelefteq U$ with $U/V \in b_U(\mathscr{H})$ and using the inductive hypothesis, we obtain

$$K/N \in \mathrm{Cov}_{\mathscr{H}}(U/N);$$

note that $K/V = KV/V \in \mathrm{Cov}_{\mathscr{H}}(U/V)$ and $K/N \in \mathrm{Cov}_{\mathscr{H}}(KV/N)$. As we shall see below, primitive-closure of $\mathscr{H}$ alone implies conjugacy of all elements of $\mathrm{Cov}_{\mathscr{H}}(U/N)$. Therefore, w.l.o.g., $K/N = HN/N$, and

$$H \in \mathrm{Cov}_{\mathscr{H}}(HN) \text{ coincides with } K = HN,$$

from which $U = (HY)N = HY$ follows. This contradiction proves our claim.

(c, d) This can be shown by means of a standard argument, for which the reader is referred to [4], §§3, 4. To obtain G-conjugacy of all covering subgroups, one has to rely on the Galois–Ore Theorem on primitive soluble groups. □

The reader should notice that the above verification of (*) relies heavily on $\mathscr{H}$ being primitive-closed as well as on solubility of G: the corresponding argument for classes of groups is more formal, in utilizing (#), and considerably easier.

2. Pronormal subgroups as covering subgroups

In this and the next section all groups will be assumed to be finite and soluble.

2.1. Proposition. *Let $H \leqq G$. Then the following four statements are equivalent in pairs:*

(i) *H is pronormal in G.*

(ii) $\mathrm{Cov}_{\hom_G(\{H\}) \cap \mathrm{Sec}(U)}(U) = \{H^u \mid u \in U\}$ *for each $U \leqq G$ such that $H \leqq U$.*

(iii) $\hom_G(\{H\})$ *is a primitive-closed homomorph.*

(iv) $\mathrm{Cov}_{\mathscr{H} \cap \mathrm{Sec}(U)}(U) = \{H^u \mid u \in U\}$ *for some G-homomorph $\mathscr{H}$ and all $U \leqq G$ containing H.*

Proof. (i) ⇒ (ii): Suppose that $H \leqq U \leqq G$. Then pronormality of H in G shows that

$$\{H^g \mid g \in G\} \cap \mathrm{Sec}(U) = \{H^u \mid u \in U\},$$

whence $\mathrm{hom}_G(\{H\}) \cap \mathrm{Sec}(U) = \mathrm{hom}_U(\{H\})$. Therefore we may prove (ii) by means of induction on $|G|$; and so it suffices to show that $G/N \in \mathrm{hom}_G(\{H\})$ for some $N \trianglelefteq G$ implies $G = HN$ — which is trivial.

(ii) $\Rightarrow$ (iii): This is trivial from 1.5a.

(iii) $\Rightarrow$ (iv): This a consequence of 1.5c, d.

(iv) $\Rightarrow$ (i): If $g \in G$ and $H, H^g \leqq U \leqq G$, then $H, H^g \in \mathrm{Cov}_{\mathscr{H} \cap \mathrm{Sec}(U)}(U)$, for $\mathscr{H} \cap \mathrm{Sec}(U) = \mathscr{H}^g \cap \mathrm{Sec}(U)$ and $H \in \mathrm{Cov}_{\mathscr{H} \cap \mathrm{Sec}(U)}(U)$ by (iv). Now $H = H^u$ for some $u \in U$ follows from (iv). □

As an example of how to apply the above characterisation of pronormal subgroups and our description of homomorphs with (conjugate) covering subgroups, we give a simple proof of Mann's characterisation of pronormality, which we then will utilize to provide short proofs of some results of Fischer (cf. [2]).

2.2. Theorem (Mann, [8]). *$H \leqq G$ is pronormal in G if, and only if, each Hall basis of G reduces into precisely one conjugate of H.*

(By a Hall basis of G we mean a complete system of Hall p'-subgroups of G.)

Proof. We shall verify the following equivalent statement:

(*) H is a pronormal subgroup in G iff for every $U \leqq G$ such that $H \leqq U$, each Hall basis of U reduces into exactly one G-conjugate of H contained in U.

"$\Rightarrow$": Assume false and let U be a minimal counterexample. Then some Hall basis Σ of U reduces into H and $H^u \neq H$ $(u \in U)$. Clearly, $U \neq H$, whence 1.2b, c and 2.1 yield a primitive quotient $U/V \in b_U(\mathrm{hom}_G(\{H\}) \cap \mathrm{Sec}(U))$, with minimal normal p-subgroup $W/V = C_{U/V}(W/V)$ say, which is complemented in U/V by both HV/V and H^uV/V. If $(U/V)_{p'}$ denotes the p'-Hall subgroup of U/V obtained from Σ, then

$$(U/V)_{p'} \leqq (HV/V) \cap (HV/V)^{uV} = C_{HV/V}(uV),$$

where we have assumed that uV was chosen in W/V. Since $(U/V)_{p'}$ contains a non-trivial normal subgroup of HV/V provided only that $HV/V \neq 1$, we conclude that $HV = H^uV$, contrary to our choice of U as a minimal counterexample.

"$\Leftarrow$": If $H \leqq U \leqq G$, then each Hall basis of U reduces into (at least) one U-conjugate of H. Hence each G-conjugate of H in U is a U-conjugate, and this is just the defining condition for H being pronormal in G. □

2.3. THEOREM (Fischer, unpublished). *Let $H_1, \dots, H_n$ be pronormal subgroups of G, put $E = \langle H_1, \dots, H_n \rangle$ and suppose that the Hall basis Σ of G reduces into each of $H_1, \dots, H_n$.*

(a) *E is pronormal in G; and Σ reduces into E.*

(b) *The E^g, $g \in G$, are precisely the minimal elements (with respect to $\leqq$) of $\mathcal{K} = \{\langle H_1^{g_1}, \dots, H_n^{g_n} \rangle \mid g_1, \dots, g_n \in G\}$.*

PROOF. (a) Let $E, E^g \leqq U \leqq G (g \in G)$ and consider a Hall basis Σ_0 of U reducing into $H_1, \dots, H_n$. Then there exists $u \in U$ such that $\Sigma_0^u \geqq \Sigma_0^g \cap U$ reduces into $H_1^g, \dots, H_n^g$. As Σ_0 can be extended to a Hall basis Σ^* of G, $H_i = H_i^{gu^{-1}}$ $(i = 1, \dots, n)$ follows from 2.2. Therefore $E^g = E^u$. In case when $U = E$ and $g = 1$, 2.2 yields $n_i \in N_G(H_i)$ $(i = 1, \dots, n)$ such that $\Sigma^* = \Sigma^{n_i}$. Hence $n_i n_1^{-1} \in N_G(\Sigma) \leqq N_G(H_i)$, $n_1 \in \bigcap_{i=1}^n N_G(H_i) \leqq N_G(E)$. Now $\Sigma = (\Sigma^*)^{n_1^{-1}}$ reduces into $E^{n_1^{-1}} = E$.

(b) Let $\Sigma_0 = \Sigma^g \cap F$ $(g \in G)$ be a Hall basis of $F = \langle H_1^{g_1}, \dots, H_n^{g_n} \rangle$, and choose $f_i \in F$ such that Σ_0 reduces into $H_i^{g_i f_i}$. From 2.2 we get that $H_i^g = H_i^{g_i f_i} \leqq F^{f_i} = F$. Consequently, $E^g = \langle H_1^g, \dots, H_n^g \rangle \leqq F$. □

2.4. COROLLARY (Fischer, unpublished). *Let the Hall basis Σ reduce into the pronormal subgroups $H_1, \dots, H_n$ of G. Suppose that $g_1, \dots, g_n \in G$ are such that $H_i^{g_i} H_j^{g_j} = H_j^{g_j} H_i^{g_i}$ $(\leqq G)$, and that Σ reduces into $H_1^{g_1} \cdots H_n^{g_n}$ $(\leqq G)$. Then $H_1^{g_1} \cdots H_n^{g_n} = \langle H_1, \dots, H_n \rangle$ is pronormal in G.*

PROOF. Put $P = H_1^{g_1} \cdots H_n^{g_n}$. Choose $p_i \in P$ $(i = 1, 2)$ such that Σ reduces into $(H_i^{g_i} H_3^{g_3} \cdots H_n^{g_n})^{p_i}$. Induction on n shows that $(H_i^{g_i} H_3^{g_3} \cdots H_n^{g_n})^{p_i} = \langle H_i, H_2, \dots, H_n \rangle$, whence $P = \langle H_1, H_3, \dots, H_n \rangle^{p_1^{-1}} \langle H_2, H_3, \dots, H_n \rangle^{p_2^{-1}}$. Therefore we may assume that $n = 2$ and $G = P = H_1^{g_1} H_2^{g_2} = H_1 H_2^g$ for some $g \in G$. W.l.o.g., $H_1 \neq G$. Induction on $|G : H_1|$ permits us to assume that $H_2 \leqq \langle H_1, H_2 \rangle = H_1$: indeed, by 2.3, $\langle H_1, H_2 \rangle$ is pronormal. Then we obtain that $G = H_1 H_2^g = H_1 H_1^g$, which is well-known to imply that $G = H_1 = \langle H_1, H_2 \rangle$. □

An immediate consequence of 2.4 is the following generalisation of a well-known result on locally pronormal subgroups.

2.5. COROLLARY. *Let $H_1, \dots, H_n$ be pronormal subgroups of G. Suppose that*

$$H_i^{g_{i,k}} H_j^{g_{j,k}} = H_j^{g_{j,k}} H_i^{g_{i,k}} \quad (g_{l,k} \in G) \qquad \text{for all } i, j \in \{1, \dots, n\}$$

and all $k \in \{1, 2\}$. Then $H_1^{g_{1,1}} \cdots H_n^{g_{n,1}}$ is conjugate in G to $H_1^{g_{1,2}} \cdots H_n^{g_{n,2}}$.

In conclusion we have two results concerning a question raised by Doerk (unpublished):

Which are the minimal non-trivial pronormal subgroups of a finite soluble group?

Rather than providing an answer, however, our first result should explain some of the difficulties to be encountered when trying to solve this problem: in fact, it turns out to be closely related to the problem of determining those Schunck classes of finite soluble groups which are minimal with respect to the order defined by strong inclusion (see [1], §§1, 2 and [3], §§2, 7); this relationship, is of course, already apparent from 2.1 above.

We shall need the following definition of *avoidance set* $a_G(\mathcal{H})$ of a G-homomorph $\mathcal{H}$; *here and in the remainder of this section we will assume that* $\mathcal{H}$ *and* $\mathcal{K}$ *denote primitive-closed* G*-homomorphs which are induced by subgroups* (so that 1.5 applies):

$$a_G(\mathcal{H}) = \{X/Y \in \mathrm{Sec}(G) \mid X/Y \text{ primitive, } H/Y \in \mathrm{Cov}_{\mathcal{H}}(X/Y) \Rightarrow H/K \text{ is contained in some complement of } F(X/Y)\}.$$

Moreover, we say that $\mathcal{H}$ is *strongly contained in* $\mathcal{K}$ ($\mathcal{H} \ll \mathcal{K}$) if, for each $U \leqq G$ and each $H \in \mathrm{Cov}_{\mathcal{H}}(U)$, there is some $K \in \mathrm{Cov}_{\mathcal{K}}(U)$ containing H; equivalently (due to conjugacy of all $\mathcal{H}$-/$\mathcal{K}$-covering subgroups), for each $U \leqq G$ and each $K \in \mathrm{Cov}_{\mathcal{K}}(U)$ there is an $H \in \mathrm{Cov}_{\mathcal{H}}(U)$ contained in K. It is a routine matter to check the following analogue of a well-known criterion for strong containment of Schunck classes (cf [1], 2.2 and [3], 2.4):

2.6. Lemma. $\mathcal{H} \ll \mathcal{K}$ *if, and only if,* $a_G(\mathcal{K}) \subseteq a_G(\mathcal{H})$.

2.7. Proposition. *Let H be a pronormal subgroup of G, and put* $\mathcal{H} = \mathrm{hom}_G(\{H\})$. *If H is a minimal non-trivial pronormal subgroup of G, then for each* $X \leqq G$, $a_X(\mathcal{H})$ *is a maximal proper avoidance set in* $\mathrm{Sec}(X)$ *or contains all primitive sections of X.*

(Here "proper" means that the set under discussion does not contain all primitive sections of X.)

Proof. The proof is straightforward from 2.6 and the relevant definitions together with the results from §1, and shall be left to the reader. □

We do not know whether a converse of 2.7 holds.

The above observation raises the questions:

Which subsets of $\mathrm{Sec}(G)$ *are avoidance sets, and which of them are maximal?*

The analogues in Schunck class theory to both of these questions are as yet unanswered, but it might, perhaps, be easier to tackle these problems in their local version.

3. On minimal pronormal subgroups: an embedding theorem

After having observed that the minimal non-trivial normally embedded subgroups of a finite soluble group G are precisely those $H \leqq G$ which are p-groups for some prime p such that $\langle H^G\rangle / O_{p'}(\langle H^G\rangle)$ is a chief factor of G covered by H, Doerk has asked if there is a simple description of the minimal non-trivial pronormal subgroups. Our aim here is to provide evidence for a negative answer to this question by showing that every finite soluble group can be embedded into a suitable minimal non-trivial pronormal subgroup of a finite soluble group.

3.1. LEMMA. (a) *Every finite soluble group is isomorphic to a subgroup of a multiprimitive finite soluble group (i.e., a group all of whose quotients are primitive).*

(b) *Let $S = CT$ be a multiprimitive group with commutator subgroup T complemented by $C \cong C_p$, and let $q \notin \pi(S)$ be a prime not dividing $p-1$.*

Then S is isomorphic to a subgroup of the twisted wreath product

$$H = T \wr_V (C_q V),$$

where $V = C \times D$ is an irreducible and faithful C_q-module over GF(p); *here $\wr_V$ refers to taking the permutation representation of $C_q V$ on $C_q V/V$, and V acts on T such that $C_V(T) = D$ and $(V/C_V(T))T \cong CT = S$.*

Moreover, H is a multiprimitive group with the following property:

if A/B is a q'-chief factor of H, then there exists a subgroup A_0/B of prime index in A/B such that $N_H(A_0/B)$ is a q'-group.

PROOF. (a) is well known. (A simple proof can be obtained by combining the Krasner–Kaloujnine Embedding Theorem with the techniques used in parts (2–4) of the proof of [5], 3.9a.)

(b) We have

$$H = (C_q V)T^{\#},$$

where $(C_q V) \cap T^{\#} = 1$, $C_q V \leqq H$, $T^{\#} = T_1 \times \cdots \times T_q \trianglelefteq H$ (with $T_i \cong T$), $CT_1 \cong CT = S$, and C_q acts on $T^{\#}$ by permuting components in the obvious way; note that for all $R \leqq T$, $R^{\#} = R_1 \times \cdots \times R_q$ (with $R_i \cong R$) denotes the "standard" subgroup of the base group $T^{\#}$ isomorphic to a direct product of $q = |C_q| = |C_q V/V|$ copies of R.

Since C_qV is clearly multiprimitive in order to establish multiprimitivity of H it will suffice to show that $A^{\#}/B^{\#}$ is a (obviously self-centralising) chief factor of H whenever A/B is a chief factor of S with $A \leqq T$. To simplify notation we may w.l.o.g. assume that $B = 1$, and then we must verify

$$\langle a^H \rangle = A^{\#} \qquad \text{for all } a \in A^{\#} \setminus \{1\}.$$

Suppose that $1 \neq (a_1, \ldots, a_q) = a \in A^{\#}$ with $a_i \in A$ $(i = 1, \ldots, q)$; w.l.o.g., $a_1 \neq 1$. Suppose that there exists $t \in T$ such that $[t, a_1] \neq 1$. Then

$$([a_1, t], 1, \ldots, 1) = [(a_1, \ldots, a_q), (t, 1, \ldots, 1)] \in \langle a^H \rangle \setminus \{1\},$$

and from irreducibility of A as a T-module and transivity of C_q on $\{1, \ldots, q\}$ it is easy to infer that $\langle a^H \rangle = A^{\#}$. Hence we are left with the case when t as above cannot be found, which in view of primitivity of S happens if, and only if, $T = A$. In this case, however, $T^{\#}$ is just the induced module $(T_{C \times D})^{C_q(C \times D)}$ with $C_{C \times D}(T) = D$. Since $q \nmid p - 1$, $D \neq 1$, whence $N_{C_q(C \times D)}(D) = C \times D$. Hence [3], 5.5b applies, yielding the desired conclusion.

Finally, $DT^{\#}/T^{\#} \neq 1$ has index p in $VT^{\#}/T^{\#}$ and satisfies $N_H(DT^{\#}/T^{\#}) = VT^{\#}$; and

$$\bar{A} \times \underbrace{A \times \cdots \times A}_{q-1} \leqq A^{\#},$$

where $|A/\bar{A}|$ is prime, has prime index in $A^{\#}$ and is not normalised by any conjugate of the Sylow q-subgroup C_q of H: in fact, if $C_q^{(t_1, \ldots, t_q)v}$ $(v \in V,\ t_i \in T)$ would normalise $\bar{A} \times A \times \cdots \times A$, then C_q would normalise

$$(\bar{A} \times A \times \cdots \times A)^{v^{-1}(t_1^{-1}, \ldots, t_q^{-1})} = \bar{A}^{v^{-1}t_1^{-1}} \times A \times \cdots \times A,$$

which is impossible with the action of C_q on $A^{\#}$ as given by the definition of $T \sim_V (C_qV)$. □

3.2. Lemma. *Let G be a finite (not necessarily soluble) group of order divisible by the prime p and let $P = P(1_G^p)$ denote the projective cover of the irreducible trivial* GF$(p)[G]$*-module 1_G^p. Then G is not pronormal in the semidirect product GP/S, S any proper submodule of the radical R of P.*

Proof. Put $\bar{T} = T + S/S$ for each submodule T of P. Choose a generator a of the module $\bar{P}$. Suppose that G is a conjugate in $\langle G, G^a \rangle \leqq G\bar{P}$ to G^a. Since $P/R \cong 1_G^p$, $G\bar{P}/\bar{R} = G\bar{R}/\bar{R} \times (\bar{P}/\bar{R})$, and so $\langle G, G^a \rangle \leqq G\bar{R}$. Hence there exists $b \in \bar{R}$ such that $G^b = G^a$, from which one deduces that $a - b \in C_{\bar{P}}(G)$, a proper submodule of $\bar{P}$ and thus contained in $\bar{R}$. Therefore $a = (a - b) + b \in \bar{R}$ cannot generate $\bar{P}$, the desired contradiction. □

3.3. Lemma (Gaschütz, unpublished; cf. [2]). *A subgroup H of a finite soluble group G with normal subgroup N is pronormal in G if, and only if, HN/N is pronormal in G/N and H is pronormal in $N_G(HN)$.*

3.4. Theorem. *For every finite soluble group S there exists a finite soluble group G with a minimal non-trivial pronormal subgroup H such that $S \cong S_0$ for some $S_0 \leqq H$.*

Proof. In view of 3.1a we may assume that S is multiprimitive. After having chosen a prime $q \notin \pi(S)$ not dividing $|S/S'|-1$ we apply 3.1b to construct a group H with properties as listed in that lemma. Being a multiprimitive group, H is a multiple semi-direct product

$$H = H_n = V_1 \cdots V_n,$$

where V_i is an irreducible and faithful module for $H_{i-1} = V_1 \cdots V_{i-1}$ over GF(p_i), $i = 2, \ldots, n$.

Proceeding by induction on i, we construct groups $K_i = H_iL_i$ such that $H_i \leqq K_i$, $L_i \trianglelefteq K_i$, $H_i \cap L_i = 1$, and $K_i = K_{i-1}P_i$ $(i = 2, \ldots, n;\ K_1 = H_1$ and $L_1 = 1)$ is a semi-direct product, where P_i is a p_i-group acted upon by K_{i-1} such that $C_{K_{i-1}}(P_i) = L_{i-1}$ and

$$P_i/\Phi(P_i) \cong_{H_{i-1}} V_i \oplus V_i^* \quad (* \text{ denotes the dual module}),$$

$$\Phi(P_i) = Z(P_i) = P_i' \cong_{H_{i-1}} 1_{H_{i-1}}^{p_i} \quad (\text{the irreducible trivial GF}(p_i)[H_{i-1}]\text{-module}),$$

and the "inverse image" of V_i, V_i^* in P_i is an elementary abelian normal subgroup R_i, R_i^*, respectively, such that $R_i^{(*)} = W_i^{(*)} \oplus Z(P_i)$ with H_{i-1}-invariant subgroup $W_i^{(*)} \cong V_i^{(*)}$ satisfying $[w, \omega] = w^\omega$ for all $w \in W_i$, $\omega \in W_i^*$; this construction is a trivial modification of the one given in [7], VI.7.21, and may in fact be obtained as a quotient of the latter one. Notice that once we are given $H_{i-1} \leqq K_{i-1}$, we obtain that (to within isomorphism) $H_i = H_{i-1}V_i = H_{i-1}W_i \leqq K_i$, and $L_i = L_{i-1}R_i^*$ yields the desired normal complement of H_i in K_i — a fact on which the above recursive definition relies.

Now let $2 \leqq i \leqq n$. By construction of H via application of 3.1b, there exists a 1-codimensional H_{i-1}-submodule U_i of V_i such that $N_{H_i}(U_i)$ is a q'-group; i.e., $N_{H_i}(U_i) \leqq V_2 \cdots V_i = H_i'$. Since $V_i \leqq N_{H_i}(U_i)$, $N_{H_i}(U_i) = N_{H_{i-1}}(U_i)V_i$, and the latter group has a normal subgroup $C_{H_{i-1}}(V_i/U_i)U_i$ such that the corresponding factor group is primitive with unique minimal normal subgroup $C_{H_{i-1}}(V_i/U_i)V_i/C_{H_{i-1}}(V_i/U_i)U_i$ of prime order p_i. It is well known that therefore $N_{H_i}(U_i)$ possesses an irreducible module M_i over GF(q) with kernel

$C_{H_{i-1}}(V_i/U_i)U_i$; note that $q \neq p_i$. Let N_i be an irreducible H_i-factor module of the induced module $((M_i)_{N_{H_i}(U_i)})^{H_i}$. By the generalised Frobenius Reciprocity Theorem and Maschke's Theorem N_i contains an $N_{H_i}(U_i)$-submodule isomorphic to M_i. In particular, as the unique minimal normal subgroup V_i of H_i acts non-trivially on M_i, N_i is faithful for H_i. Viewing N_i as a module for $K_i = H_iL_i$ with kernel L_i, we define groups G_i $(i = 1, \dots, n)$ by putting

$$G_1 = K_1 = H_1, \quad G_i = K_i(N_2 \oplus \cdots \oplus N_i) \quad (i = 2, \dots, n), \quad G = G_n,$$

where N_j is a K_i-module via $K_i = K_jQ_{j,i}$ with $K_j \leqq K_i$, $Q_{j,i} \trianglelefteq K_i$, $K_j \cap Q_{j,i} = 1$ $(j < i)$: indeed, we may take $Q_{j,i} = P_{j+1} \cdots P_i$. Note that

$$G_i = G_{i-1}(P_iN_i), \quad G_{i-1} \leqq G_i, \quad P_iN_i \trianglelefteq G_i, \quad G_{i-1} \cap P_iN_i = 1 \quad (i = 2, \dots, n)$$

and $H_i \leqq K_i \leqq G_i$ $(i = 1, \dots, n)$.

Now we prove, using induction on i, that H_i is a minimal non-trivial pronormal subgroup of G_i, a statement which is obviously true if $i = 1$. Assume that this claim holds for $i - 1$. In view of the canonical isomorphism $G_i/P_iN_i \cong G_{i-1}$, which maps $H_i = H_{i-1}V_i = H_{i-1}(P_i \cap H_i)$ onto H_{i-1}, we may apply the inductive hypothesis together with 3.3 to see that in order to obtain pronormality of H_i in G_i, it suffices to verify pronormality of $H_i = H_{i-1}V_i$ in

$$N_{G_i}(H_iP_iN_i) = N_{G_{i-1}}(H_iP_iN_i)P_iN_i = N_{G_{i-1}}(H_iP_iN_i \cap G_{i-1})P_iN_i = N_{G_{i-1}}(H_{i-1})P_iN_i.$$

From

$$G_1 = K_1 = H_1, \quad G_j = K_j(N_2 \oplus \cdots \oplus N_j) \quad (2 \leqq j \leqq n),$$

where N_k is a faithful, irreducible module for $1 \neq H_k = K_k \cap H_j$ $(2 \leqq k \leqq j \leqq n)$, we get that $N_{G_{i-1}}(H_{i-1}) = N_{K_{i-1}}(H_{i-1})$, and a similar argument, referring to the construction of K_j, gives that

$$N_{K_{i-1}}(H_{i-1}) = H_{i-1} \times Z(P_2) \times \cdots \times Z(P_{i-1}).$$

Hence we have to prove pronormality of $H_i = H_{i-1}V_i$ in $(H_{i-1} \times Z(P_2) \times \cdots \times Z(P_{i-1}))P_iN_i$. Since this is a split extension, and as N_i is faithful and irreducible for H_i, we may repeat application of 3.3 to show that pronormality of $H_iR_i^*$ in $H_iP_iN_i = H_i(R_i^* \times N_i)$ and then pronormality of H_i in $H_iR_i^*$ is sufficient to prove our claim. However, as $H_iR_i^*$ is a split extension and R_i^* is a uniserial H_i-module with $R_i^*/\mathrm{Rad}_{H_i}(R_i^*) \cong V_i^*$ and $\mathrm{Rad}_{H_i}(R_i^*) \cong 1_{H_i}^{p_i}$, H_i is evidently pronormal in this group.

Thus we are left to verify minimality of H_i as a non-trivial pronormal subgroup of G_i. Therefore, suppose that $1 \neq X < H_i$ is pronormal in G_i ; note that necessarily $i \neq 1$. Then

$$XP_iN_i/P_iN_i \leqq H_iP_iN_i/P_iN_i = H_{i-1}P_iN_i/P_iN_i$$

is pronormal in $G_i/P_iN_i = G_{i-1}P_iN_i/P_iN_i$. Minimality of $H_{i-1}P_iN_i/P_iN_i$ among the non-trivial pronormal subgroups of $G_{i-1}P_iN_i/P_iN_i$ shows that one of the following two cases is given.

Case 1: $XP_iN_i/P_iN_i = 1$

In this case $X \leqq H_i \cap P_iN_i = V_i = W_i$ and so $X \neq 1$, a non-normal subgroup of the p_i-group P_i (for $[X, W_i^*] = Z(P_i)$ by construction of P_i) cannot be pronormal in G_i, which yields the desired contradiction.

Case 2: $XP_iN_i/P_iN_i = H_iP_iN_i/P_iN_i = H_{i-1}P_iN_i/P_iN_i$

Here $H_i = XP_iN_i \cap H_i = X(P_iN_i \cap H_i) = XV_i$, from which we conclude that X is a conjugate in H_i to H_{i-1}. It remains to show that H_{i-1} is not pronormal in G_i, and this will be achieved by showing that H_{i-1} is not pronormal in $H_{i-1}N_i \leqq G_i$.

Recall that the H_i-module N_i contains an irreducible $N_{H_i}(U_i)$-submodule M_i. Since

$$C_{N_{H_{i-1}}(U_i)}(M_i) = C_{H_{i-1}}(V_i/U_i)U_i$$

by choice of M_i, there exists an irreducible V_i-submodule $\bar{M}_i$ of $(M_i)_{V_i} \leqq (N_i)_{V_i}$ such that $C_{V_i}(\bar{M}_i) = U_i$. Clifford's Theorem yields that

$$(*) \qquad N_i \cong (H(\bar{M}_i)_{T_{H_i}(\bar{M}_i)})^{H_i} \text{ (an induced module)},$$

where $H(\bar{M}_i)$ = homogeneous component of $(N_i)_{V_i}$ corresponding to $\bar{M}_i$, and $T_{H_i}(\bar{M}_i)$ = (stability group in H_i of $\bar{M}_i$) = $N_{H_i}(H(\bar{M}_i))$ acts irreducibly on $H(\bar{M}_i)$.

Note that $V_i \leqq T_{H_i}(\bar{M}_i) \leqq N_{H_i}(C_{V_i}(\bar{M}_i)) = N_{H_i}(U_i)$. Therefore from $\bar{M}_i \subseteq M_i$ we conclude that

$$H(\bar{M}_i) = \langle \bar{M}_i^{T_{H_i}(\bar{M}_i)} \rangle \subseteq \langle \bar{M}_i^{N_{H_i}(U_i)} \rangle = M_i.$$

As M_i has been constructed as an irreducible and faithful module for the primitive group

$$N_{H_i}(U_i)/C_{H_{i-1}}(V_i/U_i)U_i = (N_{H_{i-1}}(U_i)U_i/C_{H_{i-1}}(V_i/U_i)U_i)$$
$$\cdot (C_{H_{i-1}}(V_i/U_i)V_i/C_{H_{i-1}}(V_i/U_i)U_i)$$

with the first factor of this product a core-free maximal subgroup and the second the unique minimal normal subgroup, a group of prime order, we may apply (a variation of) [6], 3.4.4 to get that $(M_i)_{N_{H_{i-1}}(U_i)}$ contains an irreducible trivial $N_{H_{i-1}}(U_i)$-submodule. Now using Mackey's Theorem together with the generalised Frobenius Reriprocity Theorem and Maschke's Theorem we find an irreducible trivial submodule of $H(\bar{M}_i)_{T_{H_i}(\bar{M}_i) \cap N_{H_{i-1}}(U_i)}$; note that

$$M_i = (H(\bar{M}_i)_{N_{H_i}(H(\bar{M}_i))})^{N_{H_i}(U_i)} \quad \text{and} \quad N_{H_i}(U_i) = T_{H_i}(\bar{M}_i) N_{H_{i-1}}(U_i)$$

follows from $V_i \leqq T_{H_i}(\bar{M}_i)$. Applying Mackey's Theorem to (*) we get a decomposition

$$(N_i)_{H_{i-1}} \cong ((H(\bar{M}_i))_{T_{H_i}(\bar{M}_i) \cap H_{i-1}})^{H_{i-1}} \oplus Q$$

for some H_{i-1}-module Q. Combining the last two observations yields an H_{i-1}-submodule of $(N_i)_{H_{i-1}}$ isomorphic to the induced H_{i-1}-module $(1_T^q)^{H_{i-1}}$, where $T = T_{H_i}(\bar{M}_i) \cap H_{i-1} \leqq N_{H_{i-1}}(U_i)$. Now we recall that (in view of the construction of $H = H_n$ — and thus of H_{i-1} — by means of 3.1b) $H_{i-1}/O_{q'}(H_{i-1})$ is of order q, and $N_{H_{i-1}}(U_i)$ is a q'-group, and so is a subgroup of $O_{q'}(H_{i-1})$. Clearly, $(1_T^q)^{O_{q'}(H_{i-1})}$ has a submodule isomorphic to $1^q_{O_{q'}(H_{i-1})}$, whence $(1_T^q)^{H_{i-1}}$, and thus $(N_i)_{H_{i-1}}$ too, has a submodule isomorphic to $(1^q_{O_{q'}(H_{i-1})})^{H_{i-1}}$, which (due to $H_{i-1}/O_{q'}(H_{i-1}) \cong C_q$) is just the projective cover of $1^q_{H_{i-1}}$.

In order to show that H_{i-1} is not pronormal in $H_{i-1}N_i$, it suffices to prove that H_{i-1} is not pronormal in $H_{i-1}(1^q_{O_{q'}(H_{i-1})})^{H_{i-1}} \leqq H_{i-1}N_i$; yet this follows from 3.2. □

Notice that in the above proof we have actually shown that multiprimitive groups H subject only to fairly mild restrictions on their structure, always happen to occur as minimal non-trivial pronormal subgroups of groups $G = HA$ with normal subgroup A complementing H, where A is a direct product of elementary abelian groups.

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